

Scattering theory and the Aharonov–Bohm effect in quasiclassical physics

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Scattering of a nonrelativistic quantum-mechanical particle by an impenetrable magnetic vortex is considered. The nonvanishing transverse size of the vortex is taken into account, and the limit of short, as compared to this size, wavelengths of the scattered particle is analyzed. We show that the scattering Aharonov-Bohm effect persists in the quasiclassical limit owing to the diffraction persisting in the short-wavelength limit. As a result, the vortex flux serves as a gate for the propagation of short-wavelength, almost classical, particles. This quasiclassical effect is more feasible to experimental detection in the case when space outside the vortex is conical.

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1 Introduction

The Aharonov-Bohm effect [1] (somewhat partly anticipated earlier by Ehrenberg and Siday [2]) plays a fundamental role in modern physics. It demonstrates that quantum matter is influenced by electromagnetic field even in the case when the region of nonvanishing field strength does not overlap with the region accessible to quantum matter; the indispensable condition is that the latter region be

non-simply-connected. A particular example is a magnetic field of an infinitely long solenoid which is shielded and made impenetrable to quantum matter; such a field configuration may be denoted as an impenetrable magnetic vortex. Although nowadays the Aharonov-Bohm effect is generalized in various aspects and in different areas of modern physics, in the present paper we shall discuss its traditional formulation as of a quantum-mechanical scattering effect off an impenetrable magnetic vortex (see reviews in [3, 4, 5, 6, 7]). Even in this restricted sense, it corresponds to two somewhat different but closely related setups. The first one concerns the fringe shift in the interference pattern due to two coherent particle beams under the influence of an impenetrable magnetic vortex placed between the beams. The second one deals with scattering of a particle beam directly on an impenetrable magnetic vortex. Almost all experiments are performed in the first setup, though the second setup is more elaborate from the theoretical point of view. A direct scattering experiment involving long-wavelength (slow-moving) particles is hardly possible, but that involving short-wavelength (fast-moving) particles is quite feasible, and in the present paper we propose to perform such an experiment.

Since the Aharonov-Bohm effect is the purely quantum effect which has no analogues in classical physics ¹, it becomes evidently more manifest in the limit of long wavelengths of a scattered particle, when the wave aspects of matter are exposed to the maximal extent. As the particle wavelength decreases, the wave aspects of matter are suppressed in favour of the corpuscular ones, and therefore the persistence of the Aharonov-Bohm effect in the limit of short (as compared to the vortex thickness) wavelengths seems to be rather questionable. Actually, there is a controversy in the literature concerning this point. As it follows from [3, 4], scattering off an impenetrable magnetic vortex in the short-wavelength limit tends to classical scattering off a tube of hard core, which is independent of the enclosed magnetic flux, and thus the Aharonov-Bohm effect is extinct in this limit. On the other hand, it was already shown by Aharonov and Bohm [1] for the case of an idealized (infinitely thin) vortex that the wave function vanishes in the strictly forward direction, when the vortex flux equals a half-of-odd-integer multiple of the London flux quantum; later this result was generalized to the case of a realistic vortex of finite thickness [6] and, being independent of the value of the particle wavelength, it persists in the short-wavelength limit. Thus, this circumstance witnesses in favour of the persistence of the Aharonov-Bohm effect, since the wave function for all other values of the vortex flux is for sure nonvanishing in the forward direction.

The exclusiveness of the forward direction is of no surprise. We recall the well-known fact that the short-wavelength limit of quantum-mechanical scattering off

¹ It should be noted in this respect that, on the contrary, the geometric phase of Berry [8] has its classical analogue – Hannay’s angle [9, 10].

a hard core does not converge with the classical point-particle scattering (perfect reflection), it differs by a forward peak which is due to the Fraunhofer diffraction; the peak is increasing, as the particle wavelength is decreasing and the particle is becoming like a classical point corpuscle. Meanwhile the width of the forward peak is decreasing, and that is why the experimental detection of the peak is a rather hard task. As is noted in [11], it seems more likely that the measurable quantity is the classical cross section, although the details of this phenomenon depend on the method of measurement. On the other hand, the forward peak cannot in any way be simply ignored, because its amplitude is involved in the optical theorem, whereas the amplitude yielding classical scattering vanishes in the forward direction. Also, the forward peak contributes considerably to the total cross section, making the latter twice as large as the classical total cross section.

Thus, the quantum-mechanical scattering effects persist in the quasiclassical limit owing to the diffraction effects persisting in the short-wavelength limit. Concerning the Aharonov-Bohm effect, this conjecture is justified quantitatively in the present paper.

However, our consideration is focused mainly on the scattering Aharonov-Bohm effect in conical space. Conical space is a space which is locally flat almost everywhere with exception of a region in the form of an infinitely long tube; the metric outside the tube is given by squared length element [12, 13, 14]

$$ds^2 = (1 - \eta)^{-2} d\tilde{r}^2 + \tilde{r}^2 d\varphi^2 + dz^2 = dr^2 + r^2 d\tilde{\varphi}^2 + dz^2, \quad (1)$$

where

$$\tilde{r} = r(1 - \eta), \quad 0 < \varphi < 2\pi, \quad 0 < \tilde{\varphi} < 2\pi(1 - \eta),$$

and η is related to the curvature integrated over the transverse section of the tube, being of the same sign. Deficit angle $2\pi\eta$ is bounded from above by 2π and is unbounded from below (quantity $-2\pi\eta$ for negative η is the proficit angle that can be arbitrarily large), thus $-\infty < \eta < 1$. Conical space emerges inevitably as an outer space of a topological defect in the form of a string; such defects known as the Abrikosov-Nielsen-Olesen vortices [15, 16] arise as a consequence of phase transitions with spontaneous breakdown of gauge symmetries, when the first homotopy group of the group space of the broken symmetry group is nontrivial. Certainly, the value of η is vanishingly small for vortices in superconductors, but vortices under the name of cosmic strings [17, 18] are currently discussed in cosmology and astrophysics, and the observational data is consistent with the values of η in the range $0 < \eta < 4 \cdot 10^{-7}$ (see, e.g., [19]), although the direct evidence for the existence of cosmic strings is still lacking. In carbon nanophysics, topological defects in graphene (two-dimensional crystal of carbon atoms) correspond to nanocones with the values of η equal to positive and negative multiples of $1/6$ [20, 21]. At last, conical space may emerge in a rather

general context of contemporary condensed matter physics which operates with a variety of two-dimensional structures (thin films) made of different materials. If such a film is rolled into a cone, then one can generate quasiparticle excitations in this conically-shaped film and consider their propagation towards and through the tip. In all above setups, the problem of quantum-mechanical scattering of a nonrelativistic particle by a magnetic vortex in conical space may be relevant.

Scattering in an idealized (with the core of zero transverse size) conical space was considered by 't Hooft [22] and Jackiw et al [23, 24]; later the consideration was extended to the case of an idealized magnetic vortex placed along the axis of an idealized conical space [25]. However, in the quasiclassical limit the effects of nonzero transverse size of the core become significant. These effects were taken properly into account in [26] (see also [27, 28]), and we shall implicate the results of the latter works. Instead of using the quasiclassical WKB approximation or an analogue of the Kirchhoff approximation in optics, we shall get the quasiclassical limit directly from exact expressions for the scattering amplitude.

In the next section we consider scattering of a classical point particle by an impenetrable tube of hard core in conical space. In Section 3 we consider scattering of a quantum-mechanical nonrelativistic particle by an impenetrable tube in conical space; the quasiclassical limit is obtained. In Section 4 we solve the same problem for the case when the tube is filled with the magnetic flux lines, i.e. for the case of an impenetrable magnetic vortex. The results are discussed and summarized in Section 5, the conclusions are drawn in Section 6. Some basic provisions of quantum-mechanical scattering theory in conical space are reviewed in Appendix.

2 Classical scattering off an impenetrable tube in conical space

Scattering of a classical point particle off an impenetrable tube in Euclidean space is a well-known simple problem, the solution to which can be found in a textbook on classical mechanics (see, e.g., [29]). The incidence angle is equal to the reflection angle and is denoted in the following by χ . We consider the case when incident particles fall perpendicularly to the axis of the tube of radius r_c . Scattering angle φ is related to angle χ , $\varphi = \pi - 2\chi$, and the latter defines impact parameter $\rho = r_c \sin \chi$, see Fig. 1 where the tube is directed perpendicular to the plane of the figure and its position is indicated by the circle. When space outside the tube is conical, the relation between the angles is modified:

$$(1 - \eta)(\varphi - \varphi_\eta) = \pi - 2\chi, \quad (2)$$

where $2\pi\eta$ is the deficit angle of conical space, see (1), and φ_η is connected with the purely kinematic scattering angle for particles with impact parameters exceeding the transverse size of the tube. It is instructive to consider cases of both positive and negative values of the deficit angle here. In the case $-\infty > \eta > 0$, the trajectories of particles bypassing the tube without an impact diverge, and the region of angles $-\omega_\eta < \varphi < \omega_\eta$ (where $\omega_\eta = -\eta\pi(1-\eta)^{-1}$) can be denoted as the region of shadow, see Fig. 2. In the case $0 < \eta < 1/2$, the trajectories of particles bypassing the tube without an impact converge (and intersect), and the region of angles $-\omega_\eta < \varphi < \omega_\eta$ (where $\omega_\eta = \eta\pi(1-\eta)^{-1}$) can be denoted as the region of double image, see Fig. 3. The maximal value of angle χ , which is $\pi/2$, corresponds to the minimal value of angle φ , which is φ_η , and the latter is either positive (in space with the shadow region), or negative (in space with the double-image region):

$$\varphi_\eta = -\eta\pi(1-\eta)^{-1}. \quad (3)$$

Thus, the impact parameter for particles which undergo scattering by the tube is

$$\rho = r_c(1-\eta) \sin \chi = r_c(1-\eta) \sin \left[\frac{1}{2}(1-\eta)(\pi - \varphi) \right], \quad (4)$$

where the appearance of factor $(1-\eta)$ after r_c is due to the fact that the circumference of the tube in conical space is $2\pi r_c(1-\eta)$.

As in the case of Euclidean space ($\eta = 0$), the differential cross section is related to the derivative of the impact parameter on the scattering angle, and we get the differential cross section for incident particles in the upper half-space (with respect to the horizontal plane passing through the axis of the tube):

$$\frac{d\sigma}{d\varphi} = -\frac{d\rho}{d\varphi} = \frac{1}{2}r_c(1-\eta)^2 \sin \left[\frac{1}{2}(1-\eta)\varphi + \frac{1}{2}\eta\pi \right], \quad \varphi_\eta < \varphi < \pi. \quad (5)$$

A similar consideration yields the differential cross section for incident particles in the lower half-space:

$$\frac{d\sigma}{d\varphi} = \frac{d\rho}{d\varphi} = \frac{1}{2}r_c(1-\eta)^2 \sin \left[\frac{1}{2}(1-\eta)\varphi + \frac{1}{2}\eta\pi \right], \quad \pi < \varphi < 2\pi - \varphi_\eta. \quad (6)$$

It should be emphasized that, although expressions (5) and (6) are true for $-\infty < \eta < 1/2$, their physical consequences for negative and positive values of the deficit angle are quite different. Since the shadow region cannot be reached by incident particles, the total cross section in the case $-\infty < \eta < 0$ is

$$\sigma = \int_{\omega_\eta}^{2\pi-\omega_\eta} d\varphi \frac{d\sigma}{d\varphi} = 2r_c(1-\eta) \quad (\omega_\eta = \varphi_\eta). \quad (7)$$

On the contrary, the double-image region can be reached by incident particles from both upper and lower half-spaces. Therefore, in the case $0 < \eta < 1/2$ we get the cross section out of the double-image region,

$$\sigma_{\text{out}} = \int_{\omega_\eta}^{2\pi-\omega_\eta} d\varphi \frac{d\sigma}{d\varphi} = 2r_c(1-\eta) \cos(\eta\pi) \quad (\omega_\eta = -\varphi_\eta), \quad (8)$$

and the cross section in the double-image region which is reached from either upper or lower half-space,

$$\sigma_{\text{in}} = \int_{-\omega_\eta}^{\omega_\eta} d\varphi \frac{d\sigma}{d\varphi} = r_c(1-\eta)[1 - \cos(\eta\pi)] \quad (\omega_\eta = -\varphi_\eta). \quad (9)$$

The total cross section in this case is

$$\sigma = \sigma_{\text{out}} + 2\sigma_{\text{in}} = 2r_c(1-\eta). \quad (10)$$

Final expressions (7) and (10) for the total cross section allow one to interpret them simply as the quotient of the circumference of the tube to the half of the complete azimuthal angle (note that only half of the tube is exposed to incident particles).

In conclusion of this section we note that the case $1/2 < \eta < 1$ which includes (in turn as η increases) spaces with shadow and double-image regions can be considered as well.

3 Quasiclassical limit of quantum-mechanical scattering off an impenetrable tube in conical space

In quantum mechanics, one considers a scattering wave solution to the Schrödinger equation $i\hbar\partial_t\psi = H\psi$. In the case of a cylindrically symmetric potential, the asymptotics of this solution at large distances from the symmetry axis takes form

$$\psi \sim \exp(-iEt\hbar^{-1} + ik_z z)[\psi_{\text{in}}(\mathbf{r}; \mathbf{k}) + f(k, \varphi) \exp(ikr)r^{-1/2} + O(r^{-1})], \quad (11)$$

where \mathbf{k} is the two-dimensional wave vector which is orthogonal to the symmetry axis, $k^2 = 2mE\hbar^{-2} - k_z^2 > 0$, m and E are the mass and the energy of a scattered particle. If the potential corresponds to an impenetrable tube of radius r_c and

infinite length, then the particle wave function obeys the Dirichlet boundary condition at the edge of the tube

$$\psi|_{r=r_c} = 0. \quad (12)$$

If space outside the tube is Euclidean, then the incident wave can be a plane wave in, say, the $\varphi = 0$ direction,

$$\psi_{\text{in}}(\mathbf{r}; \mathbf{k}) = \exp(ikr \cos \varphi), \quad (13)$$

and the scattering amplitude is given by expression

$$f(k, \varphi) = -\sqrt{\frac{2}{\pi i k}} \sum_{n \in \mathbb{Z}} e^{in\varphi} \frac{J_{|n|}(kr_c)}{H_{|n|}^{(1)}(kr_c)}, \quad (14)$$

where $J_\nu(u)$ and $H_\nu^{(1)}(u)$ are the Bessel and the first-kind Hankel functions of order ν , \mathbb{Z} is the set of integer numbers.

The case of a fast-moving incident particle, $kr_c \gg 1$, corresponds to the quasiclassical limit when the laws of geometric (ray) optics seem to be applicable. One can get this limit directly from exact expression (14) which is valid for all values of k . The crucial point of the analysis is that the Bessel function becomes vanishingly small when its order exceeds its large argument and, therefore, the infinite sum in (14) is cut at $|n| \sim kr_c$ in the case $kr_c \gg 1$ (see, e.g., [30]). To be more precise, we rewrite (14) as

$$f(k, \varphi) = -\frac{1}{\sqrt{2\pi i k}} \left[\sum_{|n| \leq kr_c} e^{in\varphi} + \sum_{|n| \leq kr_c} e^{in\varphi} \frac{H_{|n|}^{(2)}(kr_c)}{H_{|n|}^{(1)}(kr_c)} + 2 \sum_{|n| > kr_c} e^{in\varphi} \frac{J_{|n|}(kr_c)}{H_{|n|}^{(1)}(kr_c)} \right]. \quad (15)$$

where $H_\nu^{(2)}(u)$ is the second-kind Hankel function of order ν . The first sum in (15) is easily taken as a sum of geometric progression, and its contribution to the scattering amplitude is

$$f^{(\text{peak})}(k, \varphi) = -\frac{1}{\sqrt{2\pi i k}} \frac{\sin[(\llbracket kr_c \rrbracket + 1/2)\varphi]}{\sin(\varphi/2)}, \quad (16)$$

where $\llbracket x \rrbracket$ denotes the integer part of quantity x (i.e. the integer which is less than or equal to x). In the case $kr_c \gg 1$, when $\llbracket kr_c \rrbracket + 1/2 \approx kr_c$, (16) is strongly peaked in the strictly forward direction, $\varphi = 0$, vanishing otherwise:

$$f^{(\text{peak})}(k, \varphi) = -\sqrt{\frac{2k}{\pi i}} r_c \{1 + O[(kr_c \varphi)^2]\}, \quad |\varphi| \ll (kr_c)^{-1}. \quad (17)$$

In view of relation

$$\frac{1}{4\pi x} \int_{-\pi}^{\pi} d\varphi \frac{\sin^2(x\varphi)}{\sin^2(\varphi/2)} = 1 + O(x^{-2}), \quad x \gg 1,$$

one can introduce function

$$\Delta_x(\varphi) = \frac{1}{4\pi x} \frac{\sin^2(x\varphi)}{\sin^2(\varphi/2)} \quad (-\pi < \varphi < \pi), \quad (18)$$

as a regularized (smoothed) delta-function for the angular variable:

$$\lim_{x \rightarrow \infty} \Delta_x(\varphi) = \Delta(\varphi) \equiv \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in\varphi}, \quad \Delta_x(0) = \frac{x}{\pi}. \quad (19)$$

Then (16) in the case $kr_c \gg 1$ is rewritten as

$$f^{(\text{peak})}(k, \varphi) = -\sqrt{-2ir_c \Delta_{kr_c}(\varphi)}. \quad (20)$$

One can find the appropriate differential cross section,

$$\frac{d\sigma^{(\text{peak})}}{d\varphi} = |f^{(\text{peak})}(k, \varphi)|^2 = 2r_c \Delta_{kr_c}(\varphi), \quad (21)$$

and the appropriate integrated cross section,

$$\sigma^{(\text{peak})} = \int_{-\pi}^{\pi} d\varphi \frac{d\sigma^{(\text{peak})}}{d\varphi} = 2r_c. \quad (22)$$

The forward peak exhibited by (21) is due to the Fraunhofer diffraction that is the diffraction in almost parallel rays. It should be noted that, since the angular delta-function can be regularized in different ways, the expressions for $f^{(\text{peak})}(k, \varphi)$ can differ from (16) (see [30, 11]), but all of them correspond to $\Delta_{kr_c}(\varphi)$ with properties (19).

The second sum in (15) requires a more elaborate analysis involving the use of the stationary phase method, and its contribution to the scattering amplitude in the case $kr_c \gg 1$ is

$$f^{(\text{class})}(k, \varphi) = -\sqrt{\frac{r_c}{2}} \sin \frac{\varphi}{2} \exp(-2ikr_c \sin \frac{\varphi}{2}) + \sqrt{r_c} O[(kr_c)^{-1/2}] \quad (0 < \varphi < 2\pi). \quad (23)$$

Unlike (20), this contribution is rather uniform in the scattering angle, smoothly vanishing in the forward direction. The appropriate differential cross section,

$$\frac{d\sigma^{(\text{class})}}{d\varphi} = |f^{(\text{class})}(k, \varphi)|^2 = \frac{r_c}{2} \sin \frac{\varphi}{2}, \quad (24)$$

describes classical scattering (elastic reflection) of a point particle. The appropriate integrated cross section,

$$\sigma^{(\text{class})} = \int_0^{2\pi} d\varphi \frac{d\sigma^{(\text{class})}}{d\varphi} = 2r_c, \quad (25)$$

coincides in its value with (22). The contribution of the third sum in (15) is estimated to be of the order $\sqrt{r_c} O[(kr_c)^{-1/6}]$, and it can be neglected in the case $kr_c \gg 1$.

Thus, the quasiclassical limit does not coincide with classical theory: geometric optics is supplemented by the Fraunhofer diffraction. The incident wave in the process of scattering is transformed into two parts: the one is reflected in all directions according to the laws of geometric optics, and the other one forms the diffraction peak in the forward direction. Each part yields a cross section equal to the diameter of the tube, see (25) and (22), and the total cross section,

$$\sigma_{\text{tot}} = \sigma^{(\text{class})} + \sigma^{(\text{peak})} = 4r_c, \quad (26)$$

is twice the classical total cross section. Namely the total cross section (as well as $f^{(\text{peak})}(k, 0)$, since $f^{(\text{class})}(k, 0) = 0$) is involved in the optical theorem which expresses the probability conservation,

$$2\sqrt{\frac{2\pi}{k}} \text{Im} [i^{-1/2} f(k, 0)] = \sigma_{\text{tot}}. \quad (27)$$

However, due to the extremely narrow width of the diffraction peak, it is rather hard to detect this peak experimentally [11].

If space outside the tube is conical, then the exact expression for the scattering amplitude is

$$\begin{aligned} f(k, \varphi) = & -\exp[2ik(r_c - \xi_c)] \sqrt{\frac{2}{\pi ik}} \sum_{n \in \mathbb{Z}} e^{in(\varphi - \pi)} \left[\frac{i}{2} \sin \left(\frac{|n|\pi}{1 - \eta} \right) + \right. \\ & \left. + \exp \left(-\frac{i|n|\pi}{1 - \eta} \right) \frac{J_{|n|(1-\eta)^{-1}}(kr_c)}{H_{|n|(1-\eta)^{-1}}^{(1)}(kr_c)} \right], \end{aligned} \quad (28)$$

where $\xi_c = \int_0^{r_c} ds$ is the geodesic radius of the tube (note that the spatial region inside the tube is characterized by nonzero curvature). The contribution of the infinite sum with the first (sine) term in square brackets yields the amplitude for scattering on a tube of zero radius [22, 23], which is negligible in the quasiclassical limit. The remaining infinite sum which contains the quotient of the cylindrical functions is analyzed along the same lines as in the case of Euclidean space. The dominant contribution is given by terms with $|n| \leq kr_c(1 - \eta)$ when $kr_c \gg 1$. The diffraction peaks are now directed at $\varphi = \mp \varphi_\eta$ with φ_η given by (3), and the scattering amplitude in the quasiclassical limit takes form

$$f(k, \varphi) = f_+^{(\text{peak})}(k, \varphi + \varphi_\eta) + f_-^{(\text{peak})}(k, \varphi - \varphi_\eta) + f^{(\text{q-class})}(k, \varphi) + \sqrt{r_c} O[(kr_c)^{-1/6}]. \quad (29)$$

Here the contribution of the Fraunhofer diffraction on the tube is given by

$$\begin{aligned} f_\pm^{(\text{peak})}(k, \varphi \pm \varphi_\eta) &= -\frac{\exp[2ik(r_c - \xi_c)]}{\sqrt{2\pi ik}} \frac{\sin\left[\frac{1}{2}kr_c(1 - \eta)(\varphi \pm \varphi_\eta)\right]}{\sin\left[\frac{1}{2}(\varphi \pm \varphi_\eta)\right]} \times \\ &\times \exp\left\{\frac{i}{2}[1 \pm kr_c(1 - \eta)](\varphi \pm \varphi_\eta)\right\}, \end{aligned} \quad (30)$$

and the contribution of the quasiclassical reflection from the tube is given by (see [26])

$$\begin{aligned} f^{(\text{q-class})}(k, \varphi) &= -\exp[2ik(r_c - \xi_c)](1 - \eta)\sqrt{\frac{r_c}{2}} \times \\ &\times \sum_l \sqrt{\cos\left[\frac{1}{2}(1 - \eta)(\varphi - \pi + 2l\pi)\right]} \exp\left\{-2ikr_c \cos\left[\frac{1}{2}(1 - \eta)(\varphi - \pi + 2l\pi)\right]\right\}, \end{aligned} \quad (31)$$

where the finite sum is over integers l satisfying condition

$$-(2\pi)^{-1}(\varphi - \varphi_\eta) < l < -(2\pi)^{-1}(\varphi + \varphi_\eta) + 1. \quad (32)$$

It should be noted that the incident wave in this case differs from (13) taking the form [22, 23]

$$\psi_{\text{in}}(\mathbf{r}; \mathbf{k}) = (1 - \eta) \sum_l \exp\{-ikr \cos[(1 - \eta)(\varphi - \pi + 2l\pi)]\}, \quad (33)$$

where the summation is over the same values of l as given by (32).

The number of terms in (31) or (33), which is defined by (32) and is denoted in the following by n_l , depends on, whether the values of the scattering angle are in the range out or in the classical shadow, see Fig. 2, or in the range out or in

the classical double image, see Fig. 3. The value of n_l outside the shadow region is odd and larger by 1 than that inside the shadow region, while the value of n_l outside the double-image region is also odd but smaller by 1 than that inside the double-image region. In the case $-\infty < \eta < 0$ we have $n_l = 1$ outside and $n_l = 0$ inside, and the differential cross section corresponding to (31) is

$$\frac{d\sigma^{(\text{q-class})}}{d\varphi} = \frac{1}{2}r_c(1-\eta)^2 \sin \left[\frac{1}{2}(1-\eta)\varphi + \frac{1}{2}\eta\pi \right], \quad \varphi_\eta < \varphi < 2\pi - \varphi_\eta, \quad (34)$$

and

$$\frac{d\sigma^{(\text{q-class})}}{d\varphi} = 0, \quad -\varphi_\eta < \varphi < \varphi_\eta. \quad (35)$$

In the case $0 < \eta < 1/2$ we have $n_l = 1$ outside and $n_l = 2$ inside, and the differential cross section corresponding to (31) is

$$\frac{d\sigma^{(\text{q-class})}}{d\varphi} = \frac{1}{2}r_c(1-\eta)^2 \sin \left[\frac{1}{2}(1-\eta)\varphi + \frac{1}{2}\eta\pi \right], \quad -\varphi_\eta < \varphi < 2\pi + \varphi_\eta, \quad (36)$$

and

$$\begin{aligned} \frac{d\sigma^{(\text{q-class})}}{d\varphi} = r_c(1-\eta)^2 \left\{ \cos \left[\frac{1}{2}(1-\eta)\varphi \right] \sin \left(\frac{1}{2}\eta\pi \right) + \right. \\ \left. + \sqrt{\sin^2 \left(\frac{1}{2}\eta\pi \right) - \sin^2 \left(\frac{1}{2}(1-\eta)\varphi \right)} \cos \left[4kr_c \sin \left(\frac{1}{2}(1-\eta)\varphi \right) \cos \left(\frac{1}{2}\eta\pi \right) \right] \right\}, \\ \varphi_\eta < \varphi < -\varphi_\eta. \end{aligned} \quad (37)$$

One can note the consistency of (34)-(36) with the result of classical theory, (5) and (6); in particular, (35) signifies that the region of the classical shadow is not accessible to a quasiclassical particle. On the contrary, the region of the classical double image exhibits itself in a way which differs from that in classical theory, compare (37) with the classical differential cross section,

$$\frac{d\sigma^{(\text{class})}}{d\varphi} = r_c(1-\eta)^2 \cos \left[\frac{1}{2}(1-\eta)\varphi \right] \sin \left(\frac{1}{2}\eta\pi \right), \quad \varphi_\eta < \varphi < -\varphi_\eta, \quad (38)$$

which is the sum of (5) and (6). In particular, we get in the strictly forward direction

$$\left. \frac{d\sigma^{(\text{q-class})}}{d\varphi} \right|_{\varphi=0} = 2 \left. \frac{d\sigma^{(\text{class})}}{d\varphi} \right|_{\varphi=0} = 2r_c(1-\eta)^2 \sin \left(\frac{1}{2}\eta\pi \right). \quad (39)$$

However, integrating either (34) and (35), or (36) and (37) over the whole range of the scattering angle $(-|\varphi_\eta| < \varphi < 2\pi - |\varphi_\eta|)^2$, one gets

$$\sigma^{(\text{q-class})} = 2r_c(1 - \eta), \quad (40)$$

which coincides with the total cross section in classical theory, see (7) and (10).

As to the Fraunhofer diffraction, the contribution of the interference between the diffraction peaks, $f_+^{(\text{peak})} \left(f_-^{(\text{peak})}\right)^* + f_-^{(\text{peak})} \left(f_+^{(\text{peak})}\right)^*$, is negligible comparing to the contribution of each peak,

$$\frac{d\sigma_{\pm}^{(\text{peak})}}{d\varphi} = |f_{\pm}^{(\text{peak})}(k, \varphi \pm \varphi_\eta)|^2 = r_c(1 - \eta)\Delta_{\frac{1}{2}kr_c(1-\eta)}(\varphi \pm \varphi_\eta). \quad (41)$$

The appropriate integrated cross section is

$$\sigma_{\pm}^{(\text{peak})} = r_c(1 - \eta). \quad (42)$$

and, thus, the total cross section,

$$\sigma_{\text{tot}} = \sigma^{(\text{q-class})} + \sigma_+^{(\text{peak})} + \sigma_-^{(\text{peak})} = 4r_c(1 - \eta), \quad (43)$$

is twice the classical total cross section.

4 Quasiclassical limit of quantum-mechanical scattering off an impenetrable magnetic vortex in conical space

We consider scattering off an impenetrable tube which is filled with magnetic field with total flux Φ . The quantum-mechanical wave function obeys condition (12) at the edge of the tube, and the Schrödinger hamiltonian outside the tube takes form

$$H = -\frac{\hbar^2}{2m} \left[\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{(1 - \eta)^2 r^2} \left(\partial_\varphi - i \frac{\Phi}{\Phi_0} \right)^2 + \partial_z^2 \right], \quad (44)$$

where $\Phi_0 = 2\pi\hbar ce^{-1}$ is the London flux quantum. The exact expression for the scattering amplitude in this case is

$$f(k, \varphi) = -\exp[2ik(r_c - \xi_c)] \sqrt{\frac{2}{\pi i k}} \sum_{n \in \mathbb{Z}} e^{in(\varphi - \pi)} \left[\frac{i}{2} \sin(\alpha_n \pi) + \exp(-i\alpha_n \pi) \frac{J_{\alpha_n}(kr_c)}{H_{\alpha_n}^{(1)}(kr_c)} \right], \quad (45)$$

² Note that the second term in figure brackets in (37) yields, upon integration, the contribution which is of order $r_c O[(kr_c)^{-3/2}]$, and thereby the first term contributes only.

where

$$\alpha_n = |n - \Phi/\Phi_0|(1 - \eta)^{-1}. \quad (46)$$

The contribution of the infinite sum with the sine term in square brackets in (45) yields the amplitude for scattering off a vortex of zero transverse size, see (A.6) in Appendix, which is negligible in the quasiclassical limit. The analysis of the remaining infinite sum which contains the quotient of the cylindrical functions yields that the dominant contribution is given by terms with $\alpha_n \leq kr_c$ in the case $kr_c \gg 1$, and the scattering amplitude in this case takes form (29), where the contribution of the Fraunhofer diffraction on the vortex is given by

$$f_{\pm}^{(\text{peak})}(k, \varphi \pm \varphi_\eta) = -\frac{\exp[2ik(r_c - \xi_c)]}{\sqrt{2\pi ik}} \frac{\sin\left[\frac{1}{2}kr_c(1 - \eta)(\varphi \pm \varphi_\eta)\right]}{\sin\left[\frac{1}{2}(\varphi \pm \varphi_\eta)\right]} \times \\ \times \exp\left\{\pm i\Phi\Phi_0^{-1}(\pi - \varphi_\eta) + i\left[\llbracket\Phi\Phi_0^{-1}\rrbracket + \frac{1}{2} \pm \frac{1}{2}kr_c(1 - \eta)\right](\varphi \pm \varphi_\eta)\right\}, \quad (47)$$

and the contribution of the quasiclassical reflection from the vortex is given by (see [26])

$$f^{(\text{q-class})}(k, \varphi) = -\exp[2ik(r_c - \xi_c)](1 - \eta)\sqrt{\frac{r_c}{2}} \sum_l \sqrt{\cos\left[\frac{1}{2}(1 - \eta)(\varphi - \pi + 2l\pi)\right]} \times \\ \times \exp\left\{i\Phi\Phi_0^{-1}(\varphi - \pi + 2l\pi) - 2ikr_c \cos\left[\frac{1}{2}(1 - \eta)(\varphi - \pi + 2l\pi)\right]\right\}, \quad (48)$$

with integers l satisfying condition (32). It should be noted that (48) is valid for $-\infty < \eta < 1$, as well as is the following expression for the incident wave in this case,

$$\psi_{\text{in}}(\mathbf{r}; \mathbf{k}) = (1 - \eta) \sum_l \exp\{-ikr \cos[(1 - \eta)(\varphi - \pi + 2l\pi)]\} \exp[i\Phi\Phi_0^{-1}(\varphi - \pi + 2l\pi)], \quad (49)$$

where the summation is over the same integers l .

Thus, the crucial point is, whether the sum over l contains more than one term. If this is the case, then, due to the interference of different terms, the differential cross section corresponding to (48) is dependent on the value of the vortex flux.

In the case $-\infty < \eta \leq 0$ the differential cross section corresponding to (48) is given by (34) and (35), and this describes classical scattering which is independent of Φ . The same is true for the differential cross section out of the double-image region in the case $0 < \eta < 1/2$, see (36). On the contrary, in the double-image region in the case $0 < \eta < 1/2$, the differential cross section corresponding to

(48) takes the form which differs both from (37) and (38):

$$\begin{aligned} \frac{d\sigma^{(\text{q-class})}}{d\varphi} &= r_c(1-\eta)^2 \left\{ \cos \left[\frac{1}{2}(1-\eta)\varphi \right] \sin \left(\frac{1}{2}\eta\pi \right) + \right. \\ &+ \sqrt{\sin^2 \left(\frac{1}{2}\eta\pi \right) - \sin^2 \left(\frac{1}{2}(1-\eta)\varphi \right)} \cos \left[2\Phi\Phi_0^{-1}\pi + 4kr_c \sin \left(\frac{1}{2}(1-\eta)\varphi \right) \cos \left(\frac{1}{2}\eta\pi \right) \right] \left. \right\}, \\ &\varphi_\eta < \varphi < -\varphi_\eta. \end{aligned} \quad (50)$$

In the forward direction in the case $\sin \left(\frac{1}{2kr_c} \right) \ll \sin \left(\frac{1}{2}\eta\pi \right)$, we get

$$\begin{aligned} \frac{d\sigma^{(\text{q-class})}}{d\varphi} &= 2r_c(1-\eta)^2 \sin \left(\frac{1}{2}\eta\pi \right) \cos^2 \left[\Phi\Phi_0^{-1}\pi + kr_c(1-\eta)\varphi \cos \left(\frac{1}{2}\eta\pi \right) \right], \\ &-(kr_c)^{-1} < (1-\eta)\varphi < (kr_c)^{-1}. \end{aligned} \quad (51)$$

In particular, in the strictly forward direction, $\varphi = 0$, we get (39) at $\Phi = n\Phi_0$ and zero at $\Phi = \left(n + \frac{1}{2}\right)\Phi_0$ ($n \in \mathbb{Z}$).

The vanishing of the quasiclassical differential cross section in the strictly forward direction, when the vortex flux equals a half-of-odd-integer multiple of the London flux quantum, is a consequence of a more general result which is valid for all wavelengths of a scattered particle. Using the exact expression for the scattering wave solution to the Schrödinger equation, see (A.7) in Appendix, one can get

$$\begin{aligned} \psi(\mathbf{r}; \mathbf{k})|_{\Phi=(n+\frac{1}{2})\Phi_0} &= 2i \exp[ik(r_c - \xi_c) + i\varphi/2] \sqrt{\frac{r}{r - r_c + \xi_c}} \times \\ &\times \sum_{n' \geq 1} \sin \left[\left(n' - \frac{1}{2} \right) \varphi \right] \exp \left[in'\pi - \frac{i(n' - 1/2)\pi}{2(1-\eta)} \right] \left[J_{(n'-1/2)(1-\eta)^{-1}}(kr) - \right. \\ &\left. - \frac{J_{(n'-1/2)(1-\eta)^{-1}}(kr_c)}{H_{(n'-1/2)(1-\eta)^{-1}}^{(1)}(kr_c)} H_{(n'-1/2)(1-\eta)^{-1}}^{(1)}(kr) \right], \end{aligned} \quad (52)$$

and, thus, wave function (52) vanishes at $\varphi = 0$ for all values of k .

The dependence of the differential cross section on the vortex flux is washed off after integration over the whole range of the scattering angle, and we get cross section (40) which is independent of the vortex flux and coincides with the classical total cross section, see (10).

As to the Fraunhofer diffraction, the differential cross section corresponding to either of two diffraction peaks is given by (41), being independent of the vortex flux. The appropriate integrated cross section is given by (42), and, hence, the total cross section is given by (43).

These results should be compared with the results for the quasiclassical limit of quantum-mechanical scattering off an impenetrable magnetic vortex in Euclidean space ($\eta = 0$). The scattering amplitude in the latter case takes form

$$f(k, \varphi) = f^{(\text{peak})}(k, \varphi) + f^{(\text{class})}(k, \varphi) + \sqrt{r_c} O[(kr_c)^{-1/6}], \quad (53)$$

where the contribution of the reflection from the vortex is given by $f^{(\text{class})}$ (23), and the contribution of the Fraunhofer diffraction on the vortex is given by

$$f^{(\text{peak})}(k, \varphi) = -\sqrt{\frac{2}{\pi i k}} \frac{\sin(\frac{1}{2}kr_c\varphi)}{\sin(\varphi/2)} \cos\left(\frac{1}{2}kr_c\varphi + \Phi\Phi_0^{-1}\pi\right) \exp\left[i\left(\llbracket\Phi\Phi_0^{-1}\rrbracket + \frac{1}{2}\right)\varphi\right]. \quad (54)$$

Thus, the reflection is consistent with classical theory, see (24), whereas the forward diffraction peak is dependent on the vortex flux, cf. (21):

$$\begin{aligned} \frac{d\sigma^{(\text{peak})}}{d\varphi} &= 2r_c \left\{ \cos(2\Phi\Phi_0^{-1}\pi) \Delta_{kr_c}(\varphi) + \right. \\ &\left. + [1 - \cos(2\Phi\Phi_0^{-1}\pi) - \sin(2\Phi\Phi_0^{-1}\pi) \sin(kr_c\varphi)] \Delta_{\frac{1}{2}kr_c}(\varphi) \right\}. \end{aligned} \quad (55)$$

Explicitly, in the forward direction we get

$$\begin{aligned} \frac{d\sigma^{(\text{peak})}}{d\varphi} &= \frac{2}{\pi} kr_c^2 \left\{ \cos^2(\Phi\Phi_0^{-1}\pi) - \frac{1}{2} kr_c\varphi \sin(2\Phi\Phi_0^{-1}\pi) - \right. \\ &\left. - \frac{1}{24} (kr_c\varphi)^2 [1 + 7 \cos(2\Phi\Phi_0^{-1}\pi)] \right\} \{1 + O[(kr_c\varphi)^2]\}, \quad |\varphi| \ll (kr_c)^{-1}. \end{aligned} \quad (56)$$

The integrated cross section, $\sigma^{(\text{peak})}$, is independent of the vortex flux and is equal to (22). The total cross section is equal to (26) that is twice the classical total cross section.

5 Results and discussion

In the present paper we have considered quantum-mechanical scattering of a nonrelativistic short-wavelength particle by an impenetrable magnetic vortex of the finite transverse size. If space outside the vortex is Euclidean, then scattering may be decomposed into the classical reflection according to the laws of geometric optics, see (24), which is smoothly distributed in all directions, and the Fraunhofer diffraction which is strongly peaked in the forward direction, see (55); the latter depends on the vortex flux with the period equal to the London flux quantum, whereas the former does not.

This picture is changed in the case when space outside the vortex is conical. If $-\infty < \eta < 1/2$, then the Fraunhofer diffraction is peaked in two directions, see (41) where φ_η is defined by (3), and is independent of the vortex flux. If $\eta < 0$, then the forward region between the diffraction peaks is the region of the classical shadow and is inaccessible to a quasiclassical particle, see (35); scattering outside the shadow region occurs as the classical reflection, see (34). If $0 < \eta < 1/2$, then the forward region between the diffraction peaks is the region of the classical double image. Although scattering outside the double-image region occurs as the classical reflection, see (36), scattering inside the double-image region differs from that in classical theory and is dependent on the vortex flux with the period equal to the London flux quantum, see (50). Hence, the Aharonov-Bohm effect persists in the quasiclassical limit for scattering angles in the range of the double-image region in conical space with $0 < \eta < 1/2$. The physical reason of this persistence is the Fresnel diffraction that is the diffraction in converging rays (whereas the Fraunhofer diffraction is the diffraction in almost parallel rays).

Thus, we can summarize that the scattering Aharonov-Bohm effect persists in the quasiclassical limit owing to the diffraction. Although the effect is invisible for the cross section integrated over the whole range of the scattering angle, the effect reveals itself for the differential cross section, i.e. for the product of the distance to the scatterer and the intensity of the scattered wave in the units of the intensity of the incident wave. In the case of a magnetic vortex in Euclidean space, the persistence of the Aharonov-Bohm effect is due to the Fraunhofer diffraction which is peaked in the forward direction. In the case of a magnetic vortex in conical space, the peak of the Fraunhofer diffraction is shifted from the forward direction and splitted into two peaks in directions which are symmetric with respect to the forward one; the contribution of each peak is independent of the vortex flux. If the forward region between two Fraunhofer-diffraction peaks is the region of the classical shadow, then the Aharonov-Bohm effect disappears in the quasiclassical limit. If the forward region between two Fraunhofer-diffraction peaks is the region of the classical double image, then the persistence of the Aharonov-Bohm effect in the quasiclassical limit is due to the Fresnel diffraction in this region.

Since a peak of the Fraunhofer diffraction is elusive to experimental measurements, it might be hard to detect the vortex flux dependence in the strictly forward direction in Euclidean space. On the contrary, the vortex flux dependence which is due to the Fresnel diffraction in conical space looks much more likely to be detectable: it is spread over the wider region in the forward direction and its amount is finite in the quasiclassical limit, compare (51) with (56). It should be noted that the optical theorem in conical space imposes no restrictions on the scattering amplitude in the strictly forward direction; instead, it involves the scattering amplitude in two strict directions of the Fraunhofer-diffraction peaks,

see (A.9) in Appendix.

Another distinction of scattering in conical space is that it depends on spin of a scattered particle: the appropriate spin connection which is dependent on η should be introduced in hamiltonian (44). In particular, for a spin-1/2 particle the results are modified in the following way (see [26]): one should change $\Phi\Phi_0^{-1}$ to $\Phi\Phi_0^{-1} \mp \frac{1}{2}\eta$, where two signs correspond to two spin states which are defined by projections of spin on the vortex axis.

If a magnetic vortex is trapped inside a superconducting shell ³, then its flux is quantized in the units of a semifluxon, i.e. half of the London flux quantum. In view of (51) we get the following relation in the quasiclassical limit when condition $\sin\left(\frac{1}{2kr_c}\right) \ll \sin\left(\frac{1}{2}\eta\pi\right)$ is satisfied:

$$\left.\frac{d\sigma^{(\text{q-class})}}{d\varphi}\right|_{\Phi=n\Phi_0} + \left.\frac{d\sigma^{(\text{q-class})}}{d\varphi}\right|_{\Phi=(n+\frac{1}{2})\Phi_0} = 2 \left.\frac{d\sigma^{(\text{class})}}{d\varphi}\right|_{\varphi=0}, \quad (1-\eta)|\varphi| < (kr_c)^{-1}, \quad (57)$$

where the classical differential cross section is given by (38). Hence, the quasiclassical limit of the Aharonov-Bohm effect in the case when the vortex flux equals an integer multiple of a semifluxon can be presented in the form

$$\left.\frac{d\sigma^{(\text{q-class})}}{d\varphi}\right|_{\Phi=n\Phi_0/2} = F(\varphi, \pm) \left.\frac{d\sigma^{(\text{class})}}{d\varphi}\right|_{\varphi=0}, \quad (1-\eta)|\varphi| < (kr_c)^{-1}, \quad (58)$$

where the upper (lower) sign in F corresponds to even (odd) n , and

$$F(\varphi, \pm) = 1 \pm \cos \left[2kr_c(1-\eta)\varphi \cos \left(\frac{1}{2}\eta\pi \right) \right] \quad (59)$$

for a spinless particle,

$$F(\varphi, \pm) = 1 \pm \cos \left[2kr_c(1-\eta)\varphi \cos \left(\frac{1}{2}\eta\pi \right) \right] \cos(\eta\pi) \quad (60)$$

for an unpolarized spin-1/2 particle,

$$\begin{aligned} F(\varphi, \pm) = 1 \pm \cos \left[2kr_c(1-\eta)\varphi \cos \left(\frac{1}{2}\eta\pi \right) \right] \cos(\eta\pi) \pm \\ \pm \sigma \sin \left[2kr_c(1-\eta)\varphi \cos \left(\frac{1}{2}\eta\pi \right) \right] \sin(\eta\pi) \end{aligned} \quad (61)$$

for a polarized spin-1/2 particle ($\sigma = \pm 1$ correspond to two polarization states).

³ Since no magnetic field can leak outside, the superconducting shell guarantees for certain that there is no overlap between the region of magnetic flux and the region which is accessible to the scattered particle, see [5, 7].

6 Conclusions

Although the Aharonov-Bohm effect is the purely quantum effect that is alien to classical physics, it persists in the quasiclassical limit owing to the diffraction persisting in the short-wavelength limit in the region of forward scattering. To be more precise, the persistence of the scattering Aharonov-Bohm effect for quasiclassical nonrelativistic particles is due to the Fraunhofer diffraction in the case when space outside the enclosed magnetic flux is Euclidean, and the Fresnel diffraction in the case when the outer space is conical. Hence, the enclosed magnetic flux serves as a gate for the propagation of short-wavelength, almost classical, particles. In the case of conical space, this quasiclassical effect which is in principle detectable depends on the particle spin, and the most efficient gate is for the propagation of spinless particles in the strictly forward direction, see (59) at $\varphi = 0$. For instance, the propagation of fast-moving electronic excitations in the bilayer graphene sample ⁴ of conical shape can be governed by the magnetic flux applied through a hole at the tip.

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Appendix: Solution to the scattering problem and S -matrix

In the problem of quantum-mechanical scattering off a vortex in conical space one encounters a situation when the interaction with a scattering centre (vortex) is not of the potential type and is even nondecreasing at large distances from the centre. Really, the interaction hamiltonian corresponding to (44) can be presented as

$$H - H \Big|_{\substack{\Phi=0 \\ \eta=0}} = v(\mathbf{r}) + v^j(\mathbf{r}) \left(-i \frac{\partial}{\partial r^j} \right) + v^{jj'}(\mathbf{r}) \left(-\frac{\partial^2}{\partial r^j \partial r^{j'}} \right), \quad (\text{A.1})$$

where $r^1 = r \cos \varphi$, $r^2 = r \sin \varphi$ ($j, j' = 1, 2$), and

$$v \sim O(r^{-2}), \quad v^j \sim O(r^{-1}) \quad \text{and} \quad v^{jj'} \sim O(1), \quad r \rightarrow \infty; \quad (\text{A.2})$$

⁴ The relevance of hamiltonian (44) rather than the Dirac hamiltonian is due to the quadratic dispersion law in this case, see [31].

note that v^j , in contrast to v and $v^{jj'}$, is a complex function (to be more precise, the imaginary part of v^j of order r^{-1} is due to nondecrease of real quantity $v^{jj'}$ in the limit $r \rightarrow \infty$). Methods of the standard scattering theory (see, e.g., [32]) can be implemented in the case of the short-range interaction, i.e. when coefficient functions v , v^j , and $v^{jj'}$ decrease faster than $O(r^{-1-\varepsilon})$ in the limit $r \rightarrow \infty$ ($\varepsilon > 0$). The uttermost advancement to the case of the long-range interaction is due, to our knowledge, to Hörmander [33] who considered a class of interactions with a long-range part which is characterized by real coefficient functions decreasing as $O(r^{-\varepsilon})$ ($0 < \varepsilon \leq 1$). As he notes in his monograph [33], "the existence of modified wave operators is proved under the weakest sufficient conditions among all those known at the present time".

Hörmander's conditions are satisfied by the interaction in the problem of scattering off a vortex in Euclidean space ($\Phi \neq 0$ and $\eta = 0$),

$$v \sim O(r^{-2}) \quad \text{and} \quad v^j \sim O(r^{-1}), \quad r \rightarrow \infty$$

(v and v^j are real, and $v^{jj'} = 0$), and, for instance, by the interaction in the problem of scattering off a Coulomb centre,

$$v \sim O(r^{-1}), \quad r \rightarrow \infty$$

(v is real, and $v^j = v^{jj'} = 0$). On the contrary, Hörmander's conditions are violated by the interaction in the problem of scattering off a vortex in conical space ($\Phi \neq 0$ and $\eta \neq 0$). Nevertheless, scattering theory can be constructed even in the last case, and this has been done in [26], basing on earlier works [22, 23, 24, 25].

In this framework involving the use of both the time-dependent scattering theory and the Lippmann-Schwinger equation, one gets the following expression for the S -matrix in the case of scattering of a nonrelativistic particle off an impenetrable vortex of transverse radius r_c :

$$S(k, \varphi, k_z; k', \varphi', k'_z) = \frac{\delta(k - k')}{\sqrt{k k'}} \frac{e^{2ik(r_c - \xi_c)}}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(\varphi - \varphi' - \pi) - i\alpha_n \pi} \times \\ \times \left[1 - 2 \frac{J_{\alpha_n}(kr_c)}{H_{\alpha_n}^{(1)}(kr_c)} \right] \delta(k_z - k'_z), \quad (\text{A.3})$$

where the final (\mathbf{k}) and initial (\mathbf{k}') two-dimensional wave vectors of the particle are written in polar variables, and α_n is given by (46). The S -matrix is decomposed as

$$S(k, \varphi, k_z; k', \varphi', k'_z) = \left[\tilde{I}(k, \varphi; k', \varphi') + \frac{\delta(k - k')}{\sqrt{k k'}} \sqrt{\frac{ik}{2\pi}} f(k, \varphi - \varphi') \right] \delta(k_z - k'_z), \quad (\text{A.4})$$

where $f(k, \varphi)$ is given by (45) and

$$\begin{aligned}\tilde{I}(k, \varphi; k', \varphi') &= \frac{\delta(k - k')}{\sqrt{kk'}} \frac{e^{2ik(r_c - \xi_c)}}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(\varphi - \varphi' - \pi)} \cos(\alpha_n \pi) = \\ &= \frac{1}{2} \frac{\delta(k - k')}{\sqrt{kk'}} e^{2ik(r_c - \xi_c)} \left\{ \exp[i\Phi\Phi_0^{-1}(\pi - \varphi_\eta)] \Delta(\varphi - \varphi' + \varphi_\eta) + \right. \\ &\quad \left. + \exp[-i\Phi\Phi_0^{-1}(\pi - \varphi_\eta)] \Delta(\varphi - \varphi' - \varphi_\eta) \right\}. \quad (\text{A.5})\end{aligned}$$

Here, all delta-functions of the angular variables, stemming from the infinite sum in (A.3), are gathered in the modified unity matrix \tilde{I} , whereas scattering amplitude f is free of delta-functions. Thus the long-range nature of interaction manifests itself in the distortion of the usual unity matrix, $I(k, \varphi; k', \varphi') = \frac{\delta(k - k')}{\sqrt{kk'}} \Delta(\varphi - \varphi')$, the latter being appropriate to the case of the short-range interaction (for instance, in the problem of scattering off an impenetrable tube in Euclidean space). The modified unity matrix contains angular delta-functions which are peaked in two directions of the Fraunhofer diffraction, $\varphi - \varphi' = \mp \varphi_\eta$. Due to this circumstance, the incident wave is also distorted, taking the form of (49).

If one neglects the transverse size of the vortex by putting $r_c = 0$, then the contribution of the cylindrical functions in (45) vanishes and the scattering amplitude takes form

$$\begin{aligned}f(k, \varphi) &= -\sqrt{\frac{i}{2\pi k}} \sum_{n \in \mathbb{Z}} e^{in(\varphi - \pi)} \sin(\alpha_n \pi) = \\ &= \frac{1}{2} \sqrt{\frac{i}{2\pi k}} \left\{ \exp[i\Phi\Phi_0^{-1}(\pi - \varphi_\eta) + i\llbracket\Phi\Phi_0^{-1}\rrbracket(\varphi + \varphi_\eta)] \left[\cot\left(\frac{1}{2}(\varphi + \varphi_\eta)\right) + i \right] - \right. \\ &\quad \left. - \exp[-i\Phi\Phi_0^{-1}(\pi - \varphi_\eta) + i\llbracket\Phi\Phi_0^{-1}\rrbracket(\varphi - \varphi_\eta)] \left[\cot\left(\frac{1}{2}(\varphi - \varphi_\eta)\right) + i \right] \right\}. \quad (\text{A.6})\end{aligned}$$

This expression is appropriate to the ultraquantum limit, i.e. to the limit of long wavelengths of a scattered particle, $k \rightarrow 0$. The amplitude in this limit diverges in two directions corresponding to the peaks of the Fraunhofer diffraction, and, therefore, the total cross section in this limit diverges as well. However, this divergence, as well as the discontinuities of the incident wave (49) at the positions of the diffraction peaks, is an artefact of the expansion in powers of $r^{-1/2}$ in (11), which is invalid at the positions of the diffraction peaks. Really, using the exact

expression for the whole wave function

$$\begin{aligned} \psi(\mathbf{r}; \mathbf{k}) = & e^{ik(r_c - \xi_c)} \sqrt{\frac{r}{r - r_c + \xi_c}} \sum_{n \in \mathbb{Z}} e^{in(\varphi - \pi) - \frac{i}{2}\alpha_n \pi} \times \\ & \times \left[J_{\alpha_n}(kr) - \frac{J_{\alpha_n}(kr_c)}{H_{\alpha_n}^{(1)}(kr_c)} H_{\alpha_n}^{(1)}(kr) \right], \end{aligned} \quad (\text{A.7})$$

one gets, in particular, in the case $0 < \eta < 1/2$ and $kr \gg 1$, $kr_c \ll 1$ (for a more general case see [26]):

$$\begin{aligned} \psi(\mathbf{r}; \mathbf{k}) = & \frac{1}{2}(1 - \eta) \exp [ikr \pm i\Phi\Phi_0^{-1}(\pi - \varphi_\eta)] + O[(kr)^{-1/2}] + \\ & + (1 - \eta) \exp [ikr \cos(2\eta\pi) \mp i\Phi\Phi_0^{-1}(\pi + \varphi_\eta)] [1 \mp ikr(1 - \eta)(\varphi \pm \varphi_\eta) \sin(2\eta\pi)] \times \\ & \times \{1 + O[(kr)^{-1/2}]\} + O\{[kr(\varphi \pm \varphi_\eta)]^2\}, \quad kr|\varphi \pm \varphi_\eta| \ll 1, \end{aligned} \quad (\text{A.8})$$

which is finite and continuous at $\varphi = \mp\varphi_\eta$.

One more manifestation of the long-range nature of interaction is that the optical theorem in the quasiclassical limit involves regularized angular delta-function (19):

$$\begin{aligned} & \cos [\Phi\Phi_0^{-1}(\pi - \varphi_\eta)] \sqrt{\frac{2\pi}{k}} \text{Im} \left\{ i^{-1/2} e^{-2ik(r_c - \xi_c)} \left[f_+^{(\text{peak})}(k, 0) + f_-^{(\text{peak})}(k, 0) \right] \right\} - \\ & - \sin [\Phi\Phi_0^{-1}(\pi - \varphi_\eta)] \sqrt{\frac{2\pi}{k}} \text{Re} \left\{ i^{-1/2} e^{-2ik(r_c - \xi_c)} \left[f_+^{(\text{peak})}(k, 0) - f_-^{(\text{peak})}(k, 0) \right] \right\} + \\ & + \frac{2\pi}{k} \Delta_{kr_c(1-\eta)}(0) = \sigma_{\text{tot}}. \end{aligned} \quad (\text{A.9})$$

In the case of a vortex in Euclidean space ($\eta = 0$), the optical theorem in the quasiclassical limit takes form

$$2 \cos(\Phi\Phi_0^{-1}\pi) \sqrt{\frac{2\pi}{k}} \text{Im} [i^{-1/2} f^{(\text{peak})}(k, 0)] + \sin^2(\Phi\Phi_0^{-1}\pi) \frac{4\pi}{k} \Delta_{kr_c}(0) = \sigma_{\text{tot}}. \quad (\text{A.10})$$

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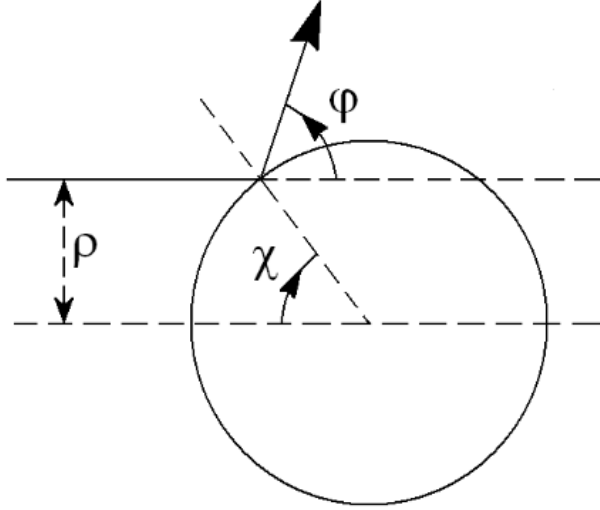


Figure 1: Scattering on a core with the incidence angle equal to the reflection angle.

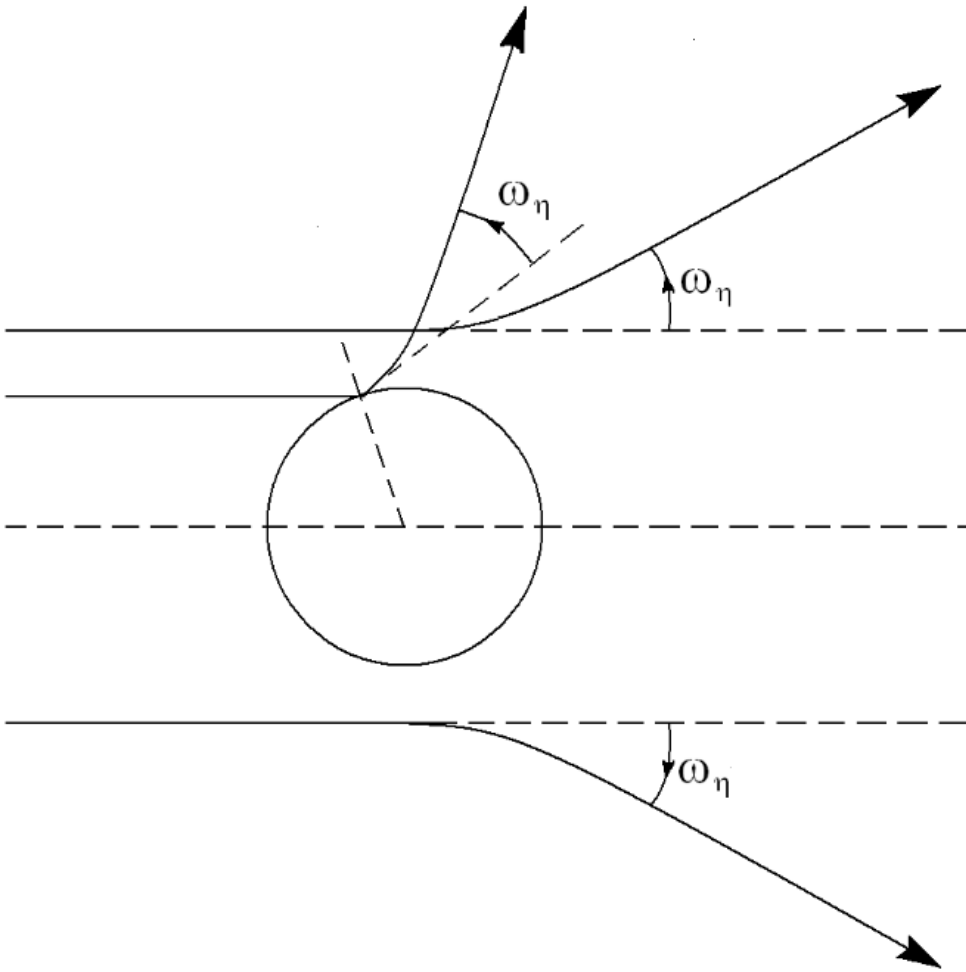


Figure 2: Scattering in conical space with $-\infty < \eta < 0$: $\omega_\eta = -\eta\pi(1 - \eta)^{-1}$.

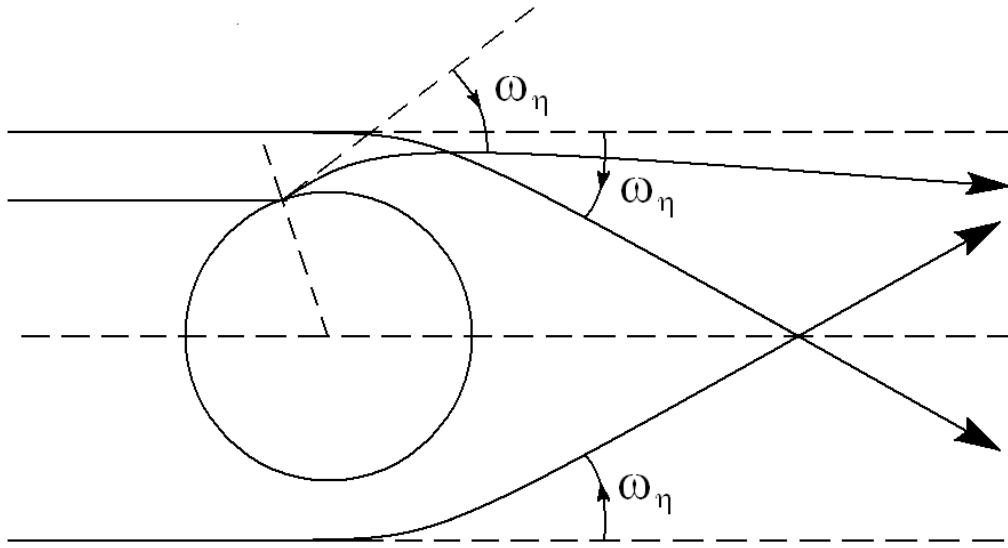


Figure 3: Scattering in conical space with $0 < \eta < 1/2$: $\omega_\eta = \eta\pi(1 - \eta)^{-1}$.